

# QUASI-INVARIANCE OF COUNTABLE PRODUCTS OF CAUCHY MEASURES UNDER TRANSLATIONS AND NON-UNITARY DILATIONS

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**ABSTRACT.** Consider an infinite sequence  $\underline{U} = (U_n)_{n=1}^\infty$  of independent Cauchy random variables, defined by a sequence  $\underline{\delta}$  of location parameters and a sequence  $\underline{\gamma}$  of scale parameters. Let  $\underline{V}$  be another infinite sequence of independent Cauchy random variables that is obtained by either additively perturbing the location parameter sequence or multiplicatively perturbing the scale parameter sequence. Using a result of Kakutani on equivalence of infinite product measures, we provide necessary and sufficient conditions for the equivalence of laws of  $\underline{U}$  and  $\underline{V}$ , and find that the Hilbert space  $\ell^2$  plays an important role in the quasi-invariance of countable products of Cauchy measures under translations and non-unitary dilations.

## 1. INTRODUCTION

Let  $\underline{U} = (U_n)_{n=1}^\infty$  be a sequence of independent, Cauchy random variables defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , where each random variable  $U_n$  is defined by the density

$$(1.1) \quad f_n(x; \delta_n, \gamma_n) = \frac{1}{\pi \gamma_n} \frac{\gamma_n^2}{(x - \delta_n)^2 + \gamma_n^2}$$

that is parametrised by the location and scale parameters  $\delta_n$  and  $\gamma_n$  respectively. We shall assume throughout that  $\gamma_n$  is strictly positive for all  $n \in \mathbb{N}$ . Define another Cauchy random variable  $V_n$  with the same scale parameter as  $U_n$ , but with location parameter given by additively perturbing  $\delta_n$  by  $h_n$ . Then the pairs of sequences  $(\underline{\delta}, \underline{\gamma})$  and  $(\underline{\delta} + \underline{h}, \underline{\gamma})$  determine the laws of  $\underline{U}$  and  $\underline{V}$  respectively. Similarly, we may define another Cauchy random variable  $W_n$  by multiplicatively perturbing the scale parameter  $\gamma_n$  by  $\sigma_n$ , and leaving the location parameter unchanged. In this case, the sequence pair  $(\underline{\delta}, \underline{\sigma\gamma})$  determines the law of  $\underline{W}$ , where  $\underline{\sigma\gamma} = (\sigma_n \gamma_n)_{n \in \mathbb{N}}$ .

In this note, we obtain necessary and sufficient conditions for the equivalence of laws of  $\underline{U}$ ,  $\underline{V}$ , and  $\underline{W}$ . We obtain these results using elementary methods, namely the Taylor series of the logarithm and results about  $\ell^p$  spaces.

In Section 2, we prove the following result concerning equivalence of laws of countable products of Cauchy measures under translations of the location parameters.

**Theorem 1.1.** *Let  $(U_n)_{n \in \mathbb{N}}$  be a sequence of independent Cauchy random variables defined by a sequence  $(\delta_n)_{n \in \mathbb{N}}$  of location parameters and a sequence  $(\gamma_n)_{n \in \mathbb{N}}$  of scale parameters. Let  $(V_n)_{n \in \mathbb{N}}$  be a sequence of independent Cauchy random variables with an additively perturbed sequence  $(\delta_n + h_n)_{n \in \mathbb{N}}$  of location parameters and the same sequence of scale parameters as  $(U_n)_{n \in \mathbb{N}}$ . Then the laws of  $(U_n)_{n \in \mathbb{N}}$  and  $(V_n)_{n \in \mathbb{N}}$  are equivalent if and only if the sequence  $(h_n/\gamma_n)_{n \in \mathbb{N}}$  of additive perturbations weighted by the scale parameters is square-summable.*

In Section 3, we prove the following result concerning equivalence of laws of countable products of Cauchy measures under non-unitary dilations.

**Theorem 1.2.** *Let  $(U_n)_{n \in \mathbb{N}}$  be a sequence of independent Cauchy random variables defined by a sequence  $(\delta_n)_{n \in \mathbb{N}}$  of location parameters and a sequence  $(\gamma_n)_{n \in \mathbb{N}}$  of scale parameters. Let  $(W_n)_{n \in \mathbb{N}}$  be a sequence of independent Cauchy random variables with a multiplicatively perturbed sequence  $(\sigma_n \gamma_n)_{n \in \mathbb{N}}$  of scale parameters and the same sequence of location parameters as  $(U_n)_{n \in \mathbb{N}}$ . Then the laws of  $(U_n)_{n \in \mathbb{N}}$  and  $(W_n)_{n \in \mathbb{N}}$  are equivalent if and only if the sequence  $(\sigma_n - 1)_{n \in \mathbb{N}}$  of differences of the multiplicative perturbations from 1 is square-summable.*

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To the best of our knowledge, the latter result appears to be new. An analogous result for the case when  $U_n$  and  $W_n$  are Gaussian is treated in [2, Example 2.7.6]. In contrast to the Gaussian case, the Cauchy case appears less straightforward, although that may be an artifact of the proof strategy we have pursued here. Both Theorem 1.1 and Theorem 1.2 establish the importance of the Hilbert space  $\ell^2$  of square-summable sequences for the equivalence of countable products of independent Cauchy random distributions.

The equivalence of laws of countable products of independent stable random variables under translations can be characterised using the Fomin calculus for measures on locally convex spaces. Let  $X$  be a locally convex space and  $\mu$  be a probability measure on a suitable  $\sigma$ -algebra of  $X$ , such as the  $\sigma$ -algebra generated by the topological dual of  $X$ . For any  $h \in X$  and  $t \in \mathbb{R}$ , define the  $th$ -shifted measure  $\mu_{th}$  by  $\mu_{th}(A) = \mu(A + th)$ , for an arbitrary measurable set  $A$ . We have the following theorem (see [1, Theorem 5.2.1]):

**Theorem 1.3.** *Let a Radon probability measure  $\mu$  on a locally convex space  $X$  be stable of some order  $\alpha \geq 1$ . Then the set  $Q(\mu)$  of all vectors  $h \in X$  such that  $\mu_{th}$  is equivalent to  $\mu_h$  for all  $t \in \mathbb{R}$  is a Hilbert space.*

The proof of Theorem 1.3 proceeds by using the fact that for  $\alpha \geq 1$ ,  $\alpha$ -stable Radon probability measures on locally convex spaces  $X$  are not only Fomin differentiable but Fomin analytic on a Hilbert subspace of  $X$ , and that analyticity implies quasi-invariance; see [1, Theorem 4.2.2]. Since  $\mathbb{R}^{\mathbb{N}}$  equipped with the countable family  $(p_n)_{n \in \mathbb{N}}$  of seminorms  $p_n(x) = |x_n|$  is a locally convex space, and since countable products of Cauchy measures are stable Radon probability measures of order  $\alpha = 1$ , then Theorem 1.3 indicates that the set of translations of the law of  $\underline{U}$  yielding an equivalent law is a Hilbert space. It follows that the sequence  $\underline{h}$  of additive perturbations of location parameters in Theorem 1.1 must lie in a space of square-summable sequences, weighted by the sequence  $\underline{\gamma}$  of scale parameters.

In this note, we prove both Theorem 1.1 and Theorem 1.2 using the formula for the probability density of the Cauchy measure, as well as Kakutani's characterisation for equivalence of product measures (see, e.g. [3, Theorem 1] and [2, Theorem 2.12.7]).

**Theorem 1.4** (Kakutani's condition for equivalence of product measures). *Let  $(X_n)_{n \in \mathbb{N}}$  and  $(Y_n)_{n \in \mathbb{N}}$  both be countable sequences of  $\mathbb{R}$ -valued independent random variables. Then the laws of  $(X_n)_{n \in \mathbb{N}}$  and  $(Y_n)_{n \in \mathbb{N}}$  are either equivalent or singular. The laws of  $(X_n)_{n \in \mathbb{N}}$  and  $(Y_n)_{n \in \mathbb{N}}$  are equivalent if and only if both the following conditions hold:*

- (i) *The laws of  $X_n$  and  $Y_n$  are equivalent for every  $n \in \mathbb{N}$ , with Radon–Nikodým derivative  $\varphi_n$ , i.e.*

$$\varphi_n(x) = \frac{d\mathbb{P} \circ Y_n^{-1}}{d\mathbb{P} \circ X_n^{-1}}(x)$$

*fpr  $\mathbb{P} \circ X_n^{-1}$ -almost all  $x$ .*

- (ii) *The series*

$$(1.2) \quad \sum_{n=1}^{\infty} -\log \mathbb{E} \left[ \sqrt{\varphi_n(X_n)} \right]$$

*converges.*

We shall also use the following lemma:

**Lemma 1.1.** *Let  $r, s \in \mathbb{N}_0$  with  $r \leq s - 1$ . Then*

$$\int_{\mathbb{R}} \frac{x^{2r}}{(x^2 + 1)^s} dx = \pi \frac{(2r)!(2(s - r - 1))!}{4^{s-1} r! (s - r - 1)! (s - 1)!}$$

*Proof.* Letting  $y := x^2$ , we have  $dx = (2\sqrt{y})^{-1} dy$ , so

$$\int_{\mathbb{R}} \frac{x^{2r}}{(x^2 + 1)^s} dx = \int_0^{\infty} \frac{y^{r-1/2}}{(y + 1)^s} dy = \int_0^{\infty} \frac{y^{(r+1/2)-1}}{(y + 1)^s} dy.$$

Using [4, Equation (2)] and properties of the gamma function  $\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x} dx$  (for  $\operatorname{Re} t > 0$ ), we have, for  $0 < \operatorname{Re} (r + 1/2) < \operatorname{Re} s$ ,

$$\int_0^{\infty} \frac{y^{(r+1/2)-1}}{(y + 1)^s} dy = \frac{\Gamma(r + \frac{1}{2}) \Gamma(s - r - \frac{1}{2})}{\Gamma(s)} = \frac{(2r)!}{4^r r!} \sqrt{\pi} \frac{(2(s - r - 1))!}{4^{s-r-1} (s - r - 1)!} \sqrt{\pi} \frac{1}{(s - 1)!}.$$

Simplifying the right-hand side yields the desired conclusion.  $\square$

## 2. ADDITIVE PERTURBATIONS OF LOCATION PARAMETERS

In this section,  $U_n$  is a Cauchy random variable whose density (1.1) is parametrised by the location parameter  $\delta_n$  and the scale parameter  $\gamma_n$ , and  $V_n$  is another Cauchy random variable whose density is parametrised by the location parameter  $\delta_n + h_n$  and the same scale parameter  $\gamma_n$  as  $U_n$ . The Radon–Nikodým derivative  $\varphi_n$  of the law of  $V_n$  with respect to the law of  $U_n$  is given by

$$(2.1) \quad \varphi_n(x) = \frac{(x - \delta_n)^2 + \gamma_n^2}{(x - \delta_n - h_n)^2 + \gamma_n^2}.$$

We shall consider necessary and sufficient conditions on the sequence  $\underline{h}$  such that the laws of  $\underline{U}$  and  $\underline{V}$  are equivalent. To simplify the analysis that follows, we make the following assumption:

**Assumption 2.1.** *For all  $n \in \mathbb{N}$ ,  $h_n \neq 0$ .*

Recall that  $\gamma_n > 0$  for every  $n \in \mathbb{N}$ . We shall adopt the following notation

$$(2.2) \quad y_n := \frac{x - \delta_n}{\gamma_n}, \quad \zeta_n := \frac{h_n}{\gamma_n}.$$

We have

$$(2.3) \quad \begin{aligned} \mathbb{E} \left[ \sqrt{\varphi_n(U_n)} \right] &= \int_{\mathbb{R}} \frac{1}{\pi \gamma_n} \frac{\gamma_n^2}{\sqrt{(x - \delta_n - h_n)^2 + \gamma_n^2} \sqrt{(x - \delta_n)^2 + \gamma_n^2}} dx \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{\sqrt{(y_n - \zeta_n)^2 + 1}} \frac{1}{\sqrt{y_n^2 + 1}} dy_n, \end{aligned}$$

where we used the formula (1.1) for the density of the law of  $U_n$  and the formula (2.1) for the Radon–Nikodým derivative in the first equation, and the definitions (2.2) in the second. We shall formulate necessary and sufficient conditions for the equivalence of laws of  $\underline{U}$  and of  $\underline{V}$  in terms of the sequence  $\underline{\zeta}$  of the additive perturbations  $h_n$  weighted by the scale parameters  $\gamma_n$ .

**2.1. Taylor expansion of the summands in Kakutani’s series.** In [3], nonnegativity of the summands in the series (1.2) follows by applying the Cauchy–Schwarz inequality to show that  $\mathbb{E}[\sqrt{\varphi_n(U_n)}] \leq 1$ . In this note, we shall use a stronger condition.

**Lemma 2.2.** *Given Assumption 2.1, it follows that*

$$(2.4) \quad 0 < -\log \mathbb{E} \left[ \sqrt{\varphi_n(U_n)} \right].$$

*Proof.* By applying Jensen’s inequality to the convex function  $x \mapsto -\sqrt{x}$ , we obtain

$$-\sqrt{\mathbb{E}[\varphi_n(U_n)]} \leq -\mathbb{E} \left[ \sqrt{\varphi_n(U_n)} \right].$$

Since  $x \mapsto -\sqrt{x}$  is nonlinear, equality holds above if and only if  $\varphi_n(U_n)$  is  $\mathbb{P}$ -almost surely constant, which holds if and only if  $h_n = 0$ . Since Assumption 2.1 excludes this possibility, and since  $\varphi_n$  is the Radon–Nikodým derivative of the law of  $V_n$  with respect to the law of  $U_n$ , the above inequality becomes the strict inequality

$$(2.5) \quad \mathbb{E} \left[ \sqrt{\varphi_n(U_n)} \right] < 1,$$

from which (2.4) follows. □

From the proof of Lemma 2.2, we expect that as  $\zeta_n$  decreases to 0, then  $\mathbb{E}[\sqrt{\varphi_n(U_n)}]$  increases to 1, which is equivalent to  $-\log \mathbb{E}[\sqrt{\varphi_n(U_n)}]$  decreasing to 0. We wish to make these observations more precise.

Up to third-order terms, the Taylor expansion of  $1/\sqrt{y_n^2 + 1}$  yields that

$$(2.6) \quad \frac{1}{\sqrt{(y_n - \zeta_n)^2 + 1}} = \frac{1}{\sqrt{y_n^2 + 1}} + \zeta_n \frac{y_n}{(y_n^2 + 1)^{3/2}} + \frac{\zeta_n^2}{2!} \frac{2y_n^2 - 1}{(y_n^2 + 1)^{5/2}} + \frac{\zeta_n^3}{3!} \frac{9y_n - 6y_n^3}{(y_n^2 + 1)^{7/2}} + O(\zeta_n^4),$$

which is valid for  $|\zeta_n| < 1$ . Substituting (2.6) into (2.3) yields

$$(2.7) \quad \mathbb{E} \left[ \sqrt{\varphi_n(U_n)} \right] = 1 + \frac{\zeta_n^2}{2!} \frac{1}{\pi} \int_{\mathbb{R}} \frac{2y_n^2 - 1}{(y_n^2 + 1)^3} dy_n + O(\zeta_n^4),$$

where we have used the fact that monomials in  $y_n$  of odd power are odd functions, and hence have vanishing integrals.

We now use Lemma 1.1 to establish the following convergence result:

**Proposition 2.3.** *Suppose that Assumption 2.1 holds, and that  $\underline{\zeta}$  converges to zero. Then it holds that for every  $C > 1/8$ , there exists a  $N(C) \in \mathbb{N}$  such that*

$$0 < 1 - \mathbb{E} \left[ \sqrt{\varphi_n(U_n)} \right] \leq C\zeta_n^2, \quad \forall n \geq N(C),$$

where  $\varphi_n$  is defined in (2.1). In particular, as  $h_n/\gamma_n$  converges to 0,  $E[\sqrt{\varphi_n(U_n)}]$  converges to 1 from below.

*Proof.* The lower bound in the statement above follows from (2.5). Setting  $s = 3$  and  $r = 0, 1$  in Lemma 1.1, and using the definition (2.2) of  $\zeta_n$ , we can calculate the value of the second-order term on the right-hand side of (2.7):

$$(2.8) \quad \frac{\zeta_n^2}{2!} \frac{1}{\pi} \int_{\mathbb{R}} \frac{2y_n^2 - 1}{(y_n^2 + 1)^3} dy_n = \frac{\zeta_n^2}{2!} \left( \frac{2!2!}{4^2 2!} - \frac{4!}{4^2 2! 2!} \right) = -\frac{\zeta_n^2}{8}.$$

Thus, we may simplify (2.7) as

$$(2.9) \quad \mathbb{E} \left[ \sqrt{\varphi_n(U_n)} \right] = 1 + \zeta_n^2 R(\zeta_n),$$

where

$$(2.10) \quad R(\zeta_n) := -\frac{1}{8} + O(\zeta_n^2) < 0$$

by (2.5). From (2.10) and (2.9), we obtain

$$1 - \mathbb{E} \left[ \sqrt{\varphi_n(U_n)} \right] = -\zeta_n^2 R(\zeta_n).$$

For arbitrary  $C > 1/8$ , there exists a  $N(C) \in \mathbb{N}$  such that  $-R(\zeta_n) \leq C$  for all  $n \geq N(C)$ . This proves the first statement. The second statement follows directly from the first.  $\square$

**2.2. Sufficient conditions for mutual singularity.** We now obtain necessary conditions for the divergence of Kakutani's series. As a first step, we consider the following

**Lemma 2.4.** *If Kakutani's series in (1.2) converges, then the sequence  $\underline{\zeta}$  converges to zero.*

*Proof.* Convergence of the series in (1.2) implies that the sequence of the summands converges to zero. Zero-convergence of the summand sequence is equivalent to  $\mathbb{E}[\sqrt{\varphi_n(U_n)}]$  converging to 1, which in turn is equivalent to zero-convergence of  $\underline{\zeta}$ , by (2.9) and (2.10).  $\square$

Given the nonnegativity of the left-hand side of (2.9), it follows from (2.5) that

$$(2.11) \quad 0 < -\zeta_n^2 R(\zeta_n) < 1.$$

Therefore, by taking the logarithm of (2.9) and using the Taylor series representation of the logarithm,

$$(2.12) \quad -\log \mathbb{E} \left[ \sqrt{\varphi_n(U_n)} \right] = \sum_{m=1}^{\infty} \frac{1}{m} \zeta_n^{2m} (-R(\zeta_n))^m.$$

Observe that Lemma 2.4 is equivalent to the statement that if  $\underline{\zeta}$  does not converge to zero, then Kakutani's series (1.2) diverges, and hence the laws of  $\underline{U}$  and  $\underline{V}$  are singular, by Theorem 1.4. From the preceding discussion, we can refine this observation further.

**Proposition 2.5.** *Suppose one of the following conditions holds:*

- (1)  $\underline{\zeta}$  does not converge to zero;
- (2)  $\underline{\zeta}$  converges to zero, and  $\underline{\zeta} \notin \ell^m$  for some  $m \geq 2$ .

*Then the laws of  $\underline{U}$  and  $\underline{V}$  are mutually singular.*

*Proof.* The sufficiency of condition (1) is given in Lemma 2.4. To prove the sufficiency of condition (2), let  $0 < \varepsilon' < 1/16$  be arbitrary. Since  $\underline{\zeta}$  converges to zero, it follows from (2.10) that there exists a  $N(\varepsilon') \in \mathbb{N}$  such that

$$(2.13) \quad -R(\zeta_n) > \frac{1}{8} - \varepsilon' > \frac{1}{16}, \quad \forall n \geq N(\varepsilon').$$

Without loss of generality, we may assume that  $N(\varepsilon') = 1$ ; this corresponds to showing that the series obtained from (1.2) by discarding the first  $N(\varepsilon')$  terms diverges. By (2.12),

$$-\log \mathbb{E} \left[ \sqrt{\varphi_n(U_n)} \right] > \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{\zeta_n}{4} \right)^{2m},$$

and hence

$$\sum_{n=1}^{\infty} -\log \mathbb{E} \left[ \sqrt{\varphi_n(U_n)} \right] > \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} \left( \frac{\zeta_n}{4} \right)^{2m} = \sum_{m=1}^{\infty} \frac{1}{4^{2m} m} \|\underline{\zeta}\|_{2m}^{2m}.$$

Thus, if  $\underline{\zeta} \notin \ell^m$  for even  $m$ , Kakutani's series diverges. Since  $\|\underline{\zeta}\|_q \leq \|\underline{\zeta}\|_p$  for  $1 \leq p \leq q < \infty$ , it follows that condition (2) implies the divergence of Kakutani's series (1.2), and hence the mutual singularity of laws by Theorem 1.4.  $\square$

*Remark 1.* Observe that in condition (2) of Proposition 2.5, the space  $\ell^1$  of absolutely summable sequences plays no role in the mutual singularity of the laws of  $\underline{U}$  and  $\underline{V}$ . This perhaps surprising property follows directly from the vanishing of the coefficient of the first-order term in the series representation (2.9), which in turn follows from the fact that the coefficient of the first-order term in (2.6) is an odd function of  $y_n$ .

The following result provides a necessary condition for equivalence of laws.

**Corollary 2.6.** *If the laws of  $\underline{U}$  and  $\underline{V}$  are equivalent, and if  $\underline{\zeta}$  converges to zero, then  $\underline{\zeta} \in \ell^m$  for all  $m \geq 2$ .*

*Proof.* By Theorem 1.4, if the laws of  $\underline{U}$  and  $\underline{V}$  are equivalent, then they are not mutually singular. By the hypothesis that  $\underline{\zeta}$  converges to zero, Proposition 2.5 implies that  $\underline{\zeta} \in \ell^m$  for all  $m \geq 2$ .  $\square$

**2.3. Sufficient conditions for equivalence.** In this section, we use the (2.12) to prove the following

**Proposition 2.7.** *If  $\underline{\zeta} \in \ell^2$ , then the laws of  $\underline{U}$  and  $\underline{V}$  are equivalent.*

*Proof.* Let  $0 < \varepsilon' < 7/8$  be arbitrary. From (2.10), and using that  $\underline{\zeta}$  converges to zero, it follows that there exists a  $N_1(\varepsilon') \in \mathbb{N}$  such that

$$(2.14) \quad -R(\zeta_n) < \frac{1}{8} + \varepsilon' < 1, \quad \forall n \geq N_1(\varepsilon').$$

We note that the strict positivity of  $-R(\zeta_n)$  for all  $n \in \mathbb{N}$  follows from (2.5) and (2.9). Since  $\underline{\zeta} \in \ell^2$ , it follows that

$$N_2 := \min \left\{ N \in \mathbb{N} \mid \sum_{n=N}^{\infty} \zeta_n^2 \leq 1 \right\}.$$

is finite. Define  $N^* := \max\{N_2, N_1(1/8)\}$ . Since  $N^*$  is finite, we may discard the first  $N^* - 1$  summands in Kakutani's series (1.2) and thus we may assume that  $N^* = 1$ . Therefore, we have

$$(2.15) \quad \|\underline{\zeta}\|_{2m}^{2m} \leq (\|\underline{\zeta}\|_2^2)^m \leq 1, \quad \forall m \in \mathbb{N}.$$

Then

$$\begin{aligned} \sum_{n=1}^{\infty} -\log \mathbb{E} \left[ \sqrt{\varphi_n(U_n)} \right] &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} \zeta_n^{2m} (-R(\zeta_n))^m < \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} \zeta_n^{2m} \frac{1}{4^m} \\ &< \sum_{m=1}^{\infty} \frac{1}{4^m} \sum_{n=1}^{\infty} \zeta_n^{2m} = \sum_{m=1}^{\infty} \frac{1}{4^m} \|\underline{\zeta}\|_{2m}^{2m} \leq \sum_{m=1}^{\infty} \frac{1}{4^m} = \frac{1}{3}, \end{aligned}$$

using (2.12), (2.14) with  $\varepsilon' = 1/8$ , the fact that  $m^{-1} \leq 1$  for  $m \in \mathbb{N}$ , the definition of the  $\ell^{2m}$ -norm on the space of infinite sequences, (2.15), and the identity  $\sum_{m \in \mathbb{N}} a^m = a/(1-a)$  for  $0 < a < 1$ . Thus the series in (1.2) converges if  $\underline{\zeta} \in \ell^2$ . By Theorem 1.4, the conclusion follows.  $\square$

The proof of Theorem 1.1 now follows by recalling the definition (2.2) of  $\zeta_n := h_n/\gamma_n$ , and by combining Corollary 2.6 and Proposition 2.7.

### 3. MULTIPLICATIVE PERTURBATIONS OF SCALE PARAMETERS

Let  $U_n$  be a Cauchy random variable whose density (1.1) is parametrised by the location parameter  $\delta_n$  and the scale parameter  $\gamma_n$ . In contrast with Section 2, however, we now perturb the scale parameters while keeping the location parameters fixed. Let  $W_n$  be another Cauchy random variable whose density is parametrised by the same location parameter  $\delta_n$  as  $U_n$  and a multiplicatively perturbed scale parameter  $\sigma_n \gamma_n$ . Provided that  $|\sigma_n| > 0$ , then the Radon–Nikodým derivative of the law of  $W_n$  with respect to the law of  $U_n$  is given by

$$(3.1) \quad \varphi(x; |\sigma_n|) = |\sigma_n| \frac{(x - \delta_n)^2 + \gamma_n^2}{(x - \delta_n)^2 + \sigma_n^2 \gamma_n^2}.$$

Observe that  $\varphi(\cdot; |\sigma_n|) \equiv 1$  if and only if  $|\sigma_n| = 1$ . Given two sequences  $\underline{\delta}$  and  $\underline{\gamma}$ , we shall consider necessary and sufficient conditions on a sequence  $\underline{\sigma} = (\sigma_n)_{n \in \mathbb{N}}$  such that the law of the sequence  $\underline{W}$  of Cauchy random variables obtained from  $\underline{\sigma}$  and (3.1) is equivalent to the law of the sequence  $\underline{U}$  of Cauchy random variables determined by the two sequences  $\underline{\delta}$  and  $\underline{\gamma}$ .

Given that the convergence of Kakutani's series (1.2) implies that  $\mathbb{E}[\sqrt{\varphi_n(U_n)}]$  converges to 1, the expression (3.1) of the Radon–Nikodým density implies that  $|\sigma_n|$  converges to 1. Define the sequence  $\underline{\tau}$  by

$$(3.2) \quad \tau_n := |\sigma_n| - 1, \quad n \in \mathbb{N}.$$

Below, we shall formulate necessary and sufficient conditions for equivalence of the laws of  $\underline{U}$  and  $\underline{W}$  in terms of the sequence  $\underline{\tau}$ . We make the following assumption on  $\underline{\tau}$ :

**Assumption 3.1.** *For some  $0 < \alpha < 1$ , it holds that  $|\tau_n| \in (0, \alpha)$  for all  $n \in \mathbb{N}$ .*

The importance of  $\alpha$  is that it imposes an upper bound on the deviation of  $|\sigma_n|$  from 1. We may assume that  $\alpha \ll 1$ , because of the aforementioned convergence of  $\underline{\sigma}$  to 1. To further simplify notation, we shall use  $K_n$  to denote the  $n$ -th summand of Kakutani's series:

$$(3.3) \quad K_n := -\log \mathbb{E} \left[ \sqrt{\varphi(U_n; |\sigma_n|)} \right] \quad \forall n \in \mathbb{N}.$$

Note that by Assumption 3.1, it follows that  $\exp(-K_n) < 1$  for all  $n \in \mathbb{N}$ .

**3.1. Taylor expansion of summands in Kakutani's series.** The next result is analogous to Lemma 2.2.

**Lemma 3.2.** *Given Assumption 3.1, it follows that  $K_n$  is strictly positive, for all  $n \in \mathbb{N}$ .*

Since the proof is exactly the same as the proof of Lemma 2.2 – with the exception that we use Assumption 3.1 instead of Assumption 2.1 – we omit the proof.

Let  $y_n$  be defined as in (2.2). Analogously to (2.3), we have

$$(3.4) \quad \exp(-K_n) = \frac{\sqrt{|\sigma_n|}}{\pi} \int_{\mathbb{R}} \frac{1}{\sqrt{y_n^2 + \sigma_n^2} \sqrt{y_n^2 + 1}} dy_n.$$

Define a function  $I : (1 - \alpha, 1) \cup (1, 1 + \alpha) \rightarrow (0, \infty)$  by

$$I(|\sigma|) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{\sqrt{y^2 + 1}} \frac{1}{\sqrt{y^2 + \sigma^2}} dy = |\sigma|^{-1/2} \mathbb{E} \left[ \sqrt{\varphi(U; |\sigma|)} \right],$$

where  $U$  is a symmetric Cauchy random variable with location parameter zero and scale parameter 1. Therefore,  $I(|\sigma|) = 1$  if and only if  $|\sigma| = 1$ , and

$$(3.5) \quad |\sigma| \neq 1 \Leftrightarrow I(|\sigma|) < |\sigma|^{-1/2}.$$

Given Assumption 3.1, the integrand above is a uniformly bounded function of  $|\sigma|$  and  $y$ . Thus, by the dominated convergence theorem, we may interchange integration and differentiation to compute the derivative of  $I$  with respect to  $|\sigma|$ . By computing the higher-order derivatives of  $(y^2 + \sigma^2)^{-1/2}$  with respect to  $|\sigma|$  and employing the same argument, the assertion holds for higher-order derivatives of  $I$  as well. Thus, if we expand  $I(|\sigma|)$  in a Taylor series about  $|\sigma| = 1$ , and if for every  $\ell \in \mathbb{N}$  we define  $a_\ell$  to be the coefficient of  $(|\sigma| - 1)^\ell$ , i.e.

$$(3.6) \quad I(|\sigma|) = \sum_{\ell=0}^{\infty} a_\ell (|\sigma| - 1)^\ell \equiv \sum_{\ell=0}^{\infty} a_\ell \tau_n^\ell,$$

then

$$(3.7) \quad a_\ell = \frac{1}{\ell!} \frac{d^\ell}{d|\sigma|^\ell} I \Big|_{|\sigma|=1} = \frac{1}{\ell!} \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{\sqrt{y^2 + 1}} \left( \frac{d^\ell}{d|\sigma|^\ell} \frac{1}{\sqrt{y^2 + \sigma^2}} \Big|_{|\sigma|=1} \right) dy.$$

Note that the  $a_\ell$  are independent of  $|\sigma|$ . We have the following

**Lemma 3.3.** *The sequence  $\underline{a}$  defined by (3.7) is bounded.*

*Proof.* We prove the statement by contradiction. For an arbitrary  $C > 0$ , there exists a  $K \in \mathbb{N}$  such that  $|a_k| > C$  for all  $k \geq K$ . By (3.7), this implies that

$$\left| \frac{d^k}{d|\sigma|^k} I \right|_{|\sigma|=1} > Ck! \quad \forall k \geq K.$$

Thus, the function  $I$  grows or decreases faster than the monomial  $C|\sigma|^k$  for all  $k \geq K$ . Consequently, we can find a  $\delta > 0$  smaller than the parameter  $\alpha$  in Assumption 3.1, such that on the interval  $1 < |\sigma| < 1 + \delta$ , either  $I \geq |\sigma|^{-1/2}$  or  $I < 0$  holds, thus contradicting either (3.5) or the strict positivity of  $I$ .  $\square$

*Remark 2.* Since  $I(1) = 1$ , it must hold that  $a_0 = 1$ . By applying Lemma 1.1 and the calculus, one can obtain the values of  $a_\ell$  for  $\ell \in \mathbb{N}$ , e.g.

$$(3.8a) \quad a_1 = \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{\sqrt{y_n^2 + 1}} \frac{(-1)}{(y_n^2 + 1)^{3/2}} dy_n = -\frac{1}{\pi} \left( \pi \frac{0! \cdot 2!}{4 \cdot 1! \cdot 1!} \right) = -\frac{1}{2},$$

$$(3.8b) \quad a_2 = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{\sqrt{y_n^2 + 1}} \frac{2 - y_n^2}{(y_n^2 + 1)^{5/2}} dy_n = \frac{1}{2\pi} \left( \frac{9\pi}{8} - \frac{\pi}{2} \right) = \frac{5}{16}.$$

Given the function  $I$  and a sequence  $\underline{a}$  satisfying Assumption 3.1, we may define a sequence  $\underline{I}$  by

$$(3.9) \quad I_n := I(|\sigma_n|) = |\sigma_n|^{-1/2} \exp(-K_n), \quad \forall n \in \mathbb{N}.$$

Given Assumption 3.1 and (3.5), it follows that  $I_n < 1$  for all  $n \in \mathbb{N}$ . Define

$$(3.10) \quad \epsilon_n := \epsilon(|\sigma_n|) := \sum_{\ell=2}^{\infty} a_\ell \tau_n^{\ell-1}, \quad n \in \mathbb{N}.$$

We relate  $\epsilon_n$ ,  $I_n$  and  $\tau_n$  to each other in the next Lemma.

**Lemma 3.4.** *Suppose that Assumption 3.1 holds. Then there exists a sufficiently small  $\alpha$  such that, given  $\tau_n$ ,  $I_n$ , and  $\epsilon_n$  defined in (3.2), (3.9), and (3.10) respectively, it holds that*

$$(3.11) \quad I_n = 1 + \left( \epsilon_n - \frac{1}{2} \right) \tau_n \quad \forall n \in \mathbb{N},$$

and there exists a strictly positive constant  $C$  such that

$$(3.12) \quad |\epsilon_n| \leq C |\tau_n| \quad \forall n \in \mathbb{N}.$$

*Proof.* The first assertion (3.11) follows from the definition (3.9) of  $I_n$ , the power series representation (3.6) of  $I(|\sigma|)$ , the definitions (3.2) and (3.10) of  $\tau_n$  and  $\epsilon_n$  respectively.

Using the definitions (3.6) and (3.10) of  $I_n$  and  $\epsilon_n$  respectively, and using the value of  $a_1$  given in (3.8a) yields

$$I_n = a_0 + a_1 \tau_n + \sum_{\ell=2}^{\infty} a_\ell \tau_n^\ell = 1 - \frac{1}{2} \tau_n + \tau_n \sum_{\ell=2}^{\infty} a_\ell \tau_n^{\ell-1} = 1 - \frac{1}{2} \tau_n + \tau_n \epsilon_n,$$

which proves (3.11). To prove (3.12), we use the definition (3.10) of  $\epsilon_n$  and observe that, by Lemma 3.3,

$$|\epsilon_n| \leq \sum_{\ell=2}^{\infty} |a_\ell| |\tau_n|^{\ell-1} = C \frac{|\tau_n|}{1 - |\tau_n|} < C \frac{|\tau_n|}{1 - \alpha},$$

where  $C$  is a suitable constant satisfying the conclusion of Lemma 3.3 and  $\alpha$  is the constant in Assumption 3.1. This completes the proof.  $\square$

An important consequence of the statement (3.11) in Lemma 3.4 is that, if  $\tau_n$  is sufficiently small, then we may represent  $\log I_n$  by its Taylor series, which yields

$$(3.13) \quad \log I_n = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} (I_n - 1)^m = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \left( \epsilon_n - \frac{1}{2} \right)^m \tau_n^m.$$

The above results comprise the prerequisites for considering necessary and sufficient conditions for equivalence of laws of  $\underline{U}$  and  $\underline{W}$ .

**3.2. Necessary and sufficient conditions for equivalence of laws.** We first consider the necessary condition for equivalence, using the series representation (3.13) for  $\log I_n$  and the definition (3.2) of the sequence  $\underline{\tau}$ .

**Proposition 3.5.** *If the laws of  $\underline{U}$  and  $\underline{W}$  are equivalent, then  $\underline{\tau} \in \ell^2$ .*

*Proof.* If the laws of  $\underline{U}$  and  $\underline{W}$  are equivalent, then by Kakutani's theorem (Theorem 1.4), the sequence  $\underline{K}$  of the summands in Kakutani's series (1.2) must converge to zero. By (3.1), (3.9), and Lemma 3.4, this implies that  $\underline{\tau}$  must converge to zero. We may assume that Assumption 3.1 holds with  $\alpha$  so small that, given a fixed and arbitrary  $\epsilon' \in (0, 1/16)$ ,

$$(3.14a) \quad K_n = - \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \tau_n^m \left[ \frac{1}{2} + \left( \epsilon_n - \frac{1}{2} \right)^m \right] \quad \forall n \in \mathbb{N},$$

$$(3.14b) \quad \tau_n^{-2} \left| K_n - \frac{1}{16} \tau_n^2 \right| < \epsilon' \quad \forall n \in \mathbb{N}.$$

The first statement (3.14a) follows by choosing  $\alpha$  so small that both  $\log |\sigma_n|$  and  $\log I_n$  agree with their Taylor series, where Lemma 3.4 guarantees the existence of such an  $\alpha$  for  $\log I_n$ . To justify the second statement, we recall that  $K_n$  is strictly positive for each  $n \in \mathbb{N}$ , and use (3.14a) to obtain

$$(3.15) \quad \begin{aligned} 0 < K_n &= -\tau_n \epsilon_n + \sum_{m=2}^{\infty} \frac{(-1)^m}{m} \tau_n^m \left[ \frac{1}{2} + \left( \epsilon_n - \frac{1}{2} \right)^m \right] \\ &= -\sum_{\ell=2}^{\infty} a_{\ell} \tau_n^{\ell} + \sum_{m=2}^{\infty} \frac{(-1)^m}{m} \tau_n^m \left[ \frac{1}{2} + \left( \epsilon_n - \frac{1}{2} \right)^m \right] \\ &= \sum_{m=2}^{\infty} \tau_n^m \left[ -a_m + \frac{(-1)^m}{m} \left( \frac{1}{2} + \left( \epsilon_n - \frac{1}{2} \right)^m \right) \right], \end{aligned}$$

where we used the definition (3.10) of  $\epsilon_n$  to obtain the second equation. Observe that

$$\frac{1}{2} + \left( \epsilon_n - \frac{1}{2} \right)^2 = \frac{1}{2} + \epsilon_n^2 - \epsilon_n + \frac{1}{4}.$$

Since the definition (3.10) of  $\epsilon_n$  shows that  $\epsilon_n$  is the sum of strictly positive powers of  $\tau_n$ , it follows that the  $\epsilon_n^2 - \epsilon_n$  term on the right-hand side does not contribute to the coefficient of  $\tau_n^2$  on the right-hand side of (3.15). Thus the coefficient of  $\tau_n^2$  on the right-hand side of (3.15) equals

$$-a_2 + \frac{1}{2} \left[ \frac{1}{2} + \frac{1}{4} \right] = \frac{1}{16}$$

since (3.8b) yields that  $a_2 = -5/16$ , and we obtain

$$K_n - \frac{1}{16} \tau_n^2 = O(\tau_n^3).$$

By making  $\alpha$  sufficiently small, we can therefore ensure that (3.14b) holds. Now we have

$$\left( \frac{1}{16} - \epsilon' \right) \tau_n^2 < \frac{1}{16} \tau_n^2 - \left| K_n - \frac{1}{16} \tau_n^2 \right| \leq \frac{1}{16} \tau_n^2 + \left( K_n - \frac{1}{16} \tau_n^2 \right) = K_n.$$

Since  $0 < \epsilon' < 1/16$ , the left-hand side is strictly positive. Summing over  $n$  yields

$$0 < \left( \frac{1}{16} - \epsilon' \right) \sum_{n=1}^{\infty} \tau_n^2 < \sum_{n=1}^{\infty} K_n < \infty.$$

This completes the proof that summability of Kakutani's series implies that  $\underline{\tau} \in \ell^2$ . □

We now show that  $\underline{\tau} \in \ell^2$  suffices for the equivalence of laws of  $\underline{U}$  and  $\underline{W}$ .

**Proposition 3.6.** *If  $\underline{\tau} \in \ell^2$ , then the laws of  $\underline{U}$  and  $\underline{W}$  are equivalent.*

*Proof.* Without loss of generality, we may assume that  $\|\underline{\tau}\|_2$  is strictly less than 1, so that

$$(3.16) \quad \sum_{\ell=2}^{\infty} \sum_{n=1}^{\infty} |\tau_n^{\ell}| = \sum_{\ell=2}^{\infty} \|\underline{\tau}\|_{\ell}^{\ell} \leq \sum_{\ell=2}^{\infty} \|\underline{\tau}\|_2^{\ell} = \frac{\|\underline{\tau}\|_2^2}{1 - \|\underline{\tau}\|_2} < \infty,$$



where we used that  $\|\underline{\tau}\|_q \leq \|\underline{\tau}\|_p$  for  $1 \leq p \leq q \leq \infty$ . Since  $\underline{\tau}$  must converge to zero, we may choose  $\alpha$  in Assumption 3.1 so small that  $|\epsilon_n| < 1/8$  for all  $n \in \mathbb{N}$ , by (3.12). Since

$$\left| \frac{1}{2} + \left( \epsilon - \frac{1}{2} \right)^m \right| \leq \begin{cases} |\epsilon_n| & m = 1 \\ \frac{1}{2} + (|\epsilon_n| + \frac{1}{2})^m & m \geq 2, \end{cases}$$

the assumption that  $|\epsilon_n| < 1/8$  implies that

$$\left| \frac{1}{2} + \left( \epsilon_n - \frac{1}{2} \right)^m \right| \leq 1 \quad \forall m \in \mathbb{N}.$$

Given these assumptions, it follows that the coefficient of  $\tau_n^m$  on the right-hand side of (3.15) satisfies

$$\left| -a_m + \frac{(-1)^m}{m} \left( \frac{1}{2} + \left( \epsilon_n - \frac{1}{2} \right)^m \right) \right| \leq C + 1$$

where we have used Lemma 3.3 to bound  $|a_m|$  by some sufficiently large, strictly positive constant  $C$ . Summing (3.15) over  $n$  and employing the bound above yields

$$\sum_{n=1}^{\infty} K_n \leq (C + 1) \sum_{n=1}^{\infty} \sum_{\ell=2}^{\infty} |\tau_n^\ell| < \infty,$$

where we used (3.16) in the last inequality, both to justify the change in the order of summation and to justify the assertion of finiteness.  $\square$

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